APPLICATION OF MULTIVARIATE PADE APPROXIMATION FOR PARTIAL DIFFERENTIAL EQUATIONS (PDE)

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Abstract:
The implementation of Multivariate Padé Approximation (MPA) was examined in this paper. Multivariate Padé Approximation (MPA) was applied to the two examples solved by Adomian's Decomposition Method (ADM). That is, power series solutions by Adomian's Decomposition, was put into Multivariate Padé series form. Thus Numerical solutions of two examples were calculated and results were presented in tables and figures.

Keywords: Partial Differential Equation(PAE), Adomian's Decomposition Method, Multivariate Padé Approximation

1. Introduction

Many powerful numerical and analytical methods have been presented. Among them, the Adomian decomposition method (ADM), differential transform method (DTM) and multivariate padé approximaton (MPA) [1-8], are relatively new approaches providing an analytical and numerical approximation to linear and nonlinear problems.

Over the last 30 years many definitions and theorems have been developed for Multivariate Padé Approximations MPA. (see [11] for a survey on Multivariate Padé approximation)

2. Adomian's decomposition method [12]

The ADM, introduced by Adomian [6-8], is suitable fort the calculation of analytical solutions and approximate solutions of linear and non-linear partial differential equations. In its simplest form , taking account of the Adomian polynomials A_n and the linear and non-linear invertible operators L and N , the non-linear term in the equation is substituted by

$$N(u) = \sum_{n=0}^{\infty} A_n$$

Provided that

$$A_n = \frac{1}{n!}\frac{d}{d\mu^n}\left[N\left(\sum_{i=0}^{\infty} \mu^i u_i \right) \right] \tag{1}$$

Before applying the inverse of the linear operator of interest to the equation. Then the second-order inhomogeneous partial differential equation is expressed in the genereal form

$$u(x,t) = f(t) + x \cdot g(t) + L_x^{-1}(h(x,t) + L_x^{-1} Nu \tag{2}$$

where $u(0,t) = f(t), u_x(0,t) = g(t)$ are the initaial conditions, $h(x,t)$ is the in homogeneous part, and L_x is the linear derivative operator with inverse integral operator L_x^{-1} according to x . Hence the solution is [12]

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = f(t) + x \cdot g(t) + L_x^{-1}\left[\sum_{n=0}^{\infty} A_n + h(x,t) \right]. \tag{3}$$

3. Multivariate Padé approximation

Consider the bivariate function $f(x,y)$ with Taylor series development

$$f(x,y) = \sum_{i,j=0}^{\infty} c_{ij} x^i y^j \tag{4}$$

around the origin. We know that a solution of unvariate Padé approximation problem for

$$f(x)=\sum_{i=0}^{\infty} c_i x^i \tag{5}$$

is given by

$$p(x)=\begin{vmatrix} \sum_{i=0}^{m} c_i x^i & x\sum_{i=0}^{m-1} c_i x^i & \cdots & x^n \sum_{i=0}^{m-n} c_i x^i \\ c_{m+1} & c_m & \cdots & c_{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n} & c_{m+n-1} & \cdots & c_m \end{vmatrix}, \quad q(x)=\begin{vmatrix} 1 & x & \cdots & x^n \\ c_{m+1} & c_m & \cdots & c_{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n} & c_{m+n-1} & \cdots & c_m \end{vmatrix} \tag{6}$$

Let us now multiply j th row in $p(x)$ and $q(x)$ by x^{j+m-1} $(j=2,...,n+1)$ and afterwards divide j th column in $p(x)$ and $q(x)$ by x^{j-1} $(j=2,...,n+1)$. This results in a multiplication of numerator and denominator by x^{mn}. Having done so, if $(D=\det D_{m,n}\neq 0)$, we get

$$\frac{p(x)}{q(x)}=\frac{\begin{vmatrix} \sum_{i=0}^{m} c_i x^i & \sum_{i=0}^{m-1} c_i x^i & \cdots & \sum_{i=0}^{m-n} c_i x^i \\ c_{m+1}x^{m+1} & c_m x^m & \cdots & c_{m+1-n}x^{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n}x^{m+n} & c_{m+n-1}x^{m+n-1} & \cdots & c_m x^m \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ c_{m+1}x^{m+1} & c_m x^m & \cdots & c_{m+1-n}x^{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n}x^{m+n} & c_{m+n-1}x^{m+n-1} & \cdots & c_m x^m \end{vmatrix}} \tag{7}$$

This quotent of determinants can also immediately be written down for a bivariate function $f(x,y)$. The sum $\sum_{i=0}^{k} c_i x^i$ shall be replaced k th partial sum of the Taylor series development of $f(x,y)$ and the expression $c_k x^k$ by an expression that contains all the terms of degree k in $f(x,y)$. Here a bivariate term $c_{ij}x^i y^j$ is said to be of degree $i+j$. If we define

$$p(x,y)=\begin{vmatrix} \sum_{i+j=0}^{m} c_{ij}x^i y^j & \sum_{i+j=0}^{m-1} c_{ij}x^i y^j & \cdots & \sum_{i+j=0}^{m-n} c_{ij}x^i y^j \\ \sum_{i+j=m+1} c_{ij}x^i y^j & \sum_{i+j=m} c_{ij}x^i y^j & \cdots & \sum_{i+j=m+1-n} c_{ij}x^i y^j \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i+j=m+n} c_{ij}x^i y^j & \sum_{i+j=m+n-1} c_{ij}x^i y^j & \cdots & \sum_{i+j=m} c_{ij}x^i y^j \end{vmatrix} \tag{8}$$

and

$$q(x,y)=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \sum_{i+j=m+1} c_{ij}x^i y^j & \sum_{i+j=m} c_{ij}x^i y^j & \cdots & \sum_{i+j=m+1-n} c_{ij}x^i y^j \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i+j=m+n} c_{ij}x^i y^j & \sum_{i+j=m+n-1} c_{ij}x^i y^j & \cdots & \sum_{i+j=m} c_{ij}x^i y^j \end{vmatrix} \tag{9}$$

Then it is easy to see that $p(x,y)$ and $q(x,y)$ are of the form

$$\begin{aligned} p(x,y) &= \sum_{i+j=mn}^{mn+m} a_{ij}x^i y^j \\ q(x,y) &= \sum_{i+j=mn}^{mn+n} b_{ij}x^i y^j \end{aligned} \tag{10}$$

We know that $p(x,y)$ and $q(x,y)$ are called Padé equations[13]. So the multivariate Padé approximant of order (m,n) for $f(x,y)$ is defined as [13]

$$r_{m,n}(x,y)=\frac{p(x,y)}{q(x,y)} \tag{11}$$

4. Applications and results

In this section we consider two examples that demonstrate the performance and efficiency of the multivariate Padé approximation for solving partial differential equations.

Example 4.1.

Consider the first-order quasi-linear homogeneous partial differential equation

$$\frac{\partial u}{\partial t}+(1+t)\cdot\frac{\partial u}{\partial x}=0 \qquad (12)$$

with initial conditions

$$u(x,0)=\frac{x-1}{2}. \qquad (13)$$

Equation (12) has been solved by using ADM and the following solution has been obtained in [12]

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)=-\frac{1}{2}+\frac{1}{2}x-\frac{1}{4}t-\frac{1}{4}xt+\frac{1}{8}t^2+\frac{1}{8}xt^2-\frac{1}{16}t^3-\frac{1}{16}xt^3$$

$$+\frac{1}{32}t^4+\frac{1}{32}xt^4-\frac{1}{64}t^5-\frac{1}{64}xt^5+\cdots \qquad (14)$$

and the analytical solution is given in [12] as;

$$u(x,t)=\frac{x-t-1}{t+2}. \qquad (15)$$

Now let us calculate the approximate solution of Eq.(14) for $m=3$ and $n=1$ by using Multivariate Padé approximation. To obtain Multivariate Padé equations of Eq.(14) for $m=3$ and $n=1$, we use Eqs.(8) and (9). By using Eqs.(8) and (9) We obtain,

$$p(x,t)=\begin{vmatrix} -\frac{1}{2}+\frac{1}{2}x-\frac{1}{4}t-\frac{1}{4}xt+\frac{1}{8}t^2+\frac{1}{8}xt^2-\frac{1}{16}t^3 & -\frac{1}{2}+\frac{1}{2}x-\frac{1}{4}t-\frac{1}{4}tx+\frac{1}{8}t^2 & -\frac{1}{2}+\frac{1}{2}x-\frac{1}{4}t \\ -\frac{1}{16}xt^3+\frac{1}{32}t^4 & \frac{1}{8}xt^2-\frac{1}{16}t^3 & -\frac{1}{4}xt+\frac{1}{8}t^2 \\ -\frac{1}{32}t^4x-\frac{1}{64}t^5 & -\frac{1}{16}xt^3+\frac{1}{32}t^4 & \frac{1}{8}xt^2-\frac{1}{16}t^3 \end{vmatrix}$$

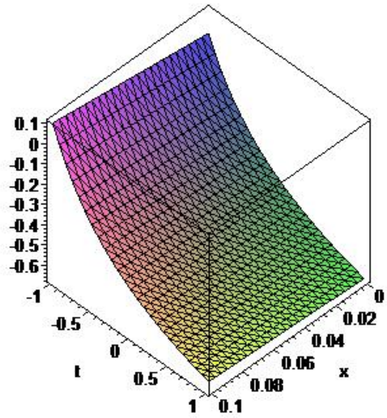
$$=\frac{t^5x(t+1-x)(-2x+t)}{256}$$

and

$$q(x,t)=\begin{vmatrix} 1 & 1 & 1 \\ -\frac{1}{16}xt^3+\frac{1}{32}t^4 & \frac{1}{8}xt^2-\frac{1}{16}t^3 & -\frac{1}{4}xt+\frac{1}{8}t^2 \\ -\frac{1}{32}t^4x-\frac{1}{64}t^5 & -\frac{1}{16}xt^3+\frac{1}{32}t^4 & \frac{1}{8}xt^2-\frac{1}{16}t^3 \end{vmatrix}=-\frac{t^5x(t+2)(-2x+t)}{256}$$

So the Multivariate Padé approximation of order $(3,1)$ for eq.(14), that is,

$$r_{3,1}(x,t)=-\frac{t+1-x}{t+2} \tag{16}$$



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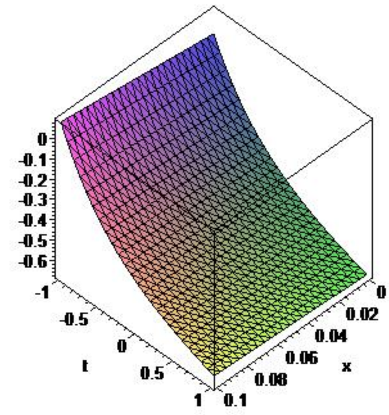

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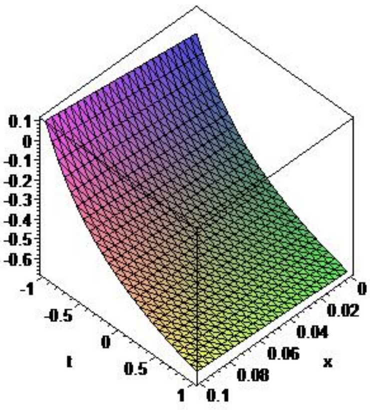

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Figures **(1)** Exact solution of partial differential equation in example1 **(2)** Adomian's decomposition solution of partial differential equation in Example 1. **(3)** Multivariate Padé approximation for Adomian's decomposition solution of partial differential equation in Example 1

Table 1. Comparison of ADM and MPA for example 1.

x	t	Exact solution	Approximate solution with ADM	Approximate solution with MPA	Absolute error of ADM	Absolute error of MPA
2.0	0.0	0.5000000000	0.5000000000	0.5000000000	0.00000000000	0.00000000000
1.9	0.1	0.3809523810	0.3809523593	0.3809523810	0.216250000×10^{-7}	0.00000000000
1.8	0.2	0.2727272727	0.2727260000	0.2727272727	$0.1272700000\times10^{-5}$	0.00000000000
1.7	0.3	0.1739130435	0.1738996719	0.1739130435	0.00001337162500	0.00000000000
1.6	0.4	0.08333333333	0.08326400000	0.08333333333	0.0000693333000	0.00000000000
1.5	0.5	0.00000000000	–0.0002441406250	0.00000000000	0.0002441406250	0.00000000000
1.4	0.6	–0.07692307692	–0.07759600000	–0.07692307692	0.000672923100	0.00000000000
1.3	0.7	–0.1481481481	–0.149714078781	–0.1481481481	0.001565930025	0.00000000000
1.2	0.8	–0.2142857143	–0.2175040000	–0.2142857143	0.003218285700	0.00000000000
1.1	0.9	–0.2758620690	–0.2818751407	–0.2758620690	0.006013071630	0.00000000000
1.0	1.0	–0.3333333333	–0.3437500000	–0.3333333333	0.01041666667	0.00000000000

Example 4.2.

Consider the partial differential equation

$$u_{xx} - u \cdot u_{tt} = 1 - \frac{x^2 + t^2}{2} \qquad (17)$$

with conditions

$$u(0,t) = \frac{t^2}{2}, \quad u_x(0.t) = 0. \qquad (18)$$

Equation (17) has been solved by using ADM and the following solution has been obtained in [12]

$$u(x,t) = \frac{1}{2}x^2 + \frac{1}{2}t^2 - \frac{1}{144}x^6t^2 - \frac{13}{10080}x^8 + \frac{1}{1120}x^8t^2 + \frac{173}{1209600}x^{10} - \frac{1}{21600}x^{10}t^2 - \frac{43}{10644480}x^{12}. \qquad (19)$$

The exact solution of equation (18) is given in [12] as

$$u(x,t)=\frac{1}{2}x^2+\frac{1}{2}t^2$$

Now let us calculate the approximate solution of Eq.(19) for $m=10$ and $n=2$ by using Multivariate Padé approximation. To obtain Multivariate Padé equations of Eq.(19) for $m=10$ and $n=2$, we use Eqs.(8) and (9). By using Eqs.(8) and (9) We obtain,

$$p(x,t)=\begin{vmatrix} \frac{1}{2}x^2+\frac{1}{2}t^2-\frac{1}{144}x^6t^2-\frac{13}{10080}x^8+\frac{1}{1120}x^8t^2+\frac{173}{1209600}x^{10} & \frac{1}{2}x^2+\frac{1}{2}t^2-\frac{1}{144}x^6t^2-\frac{13}{10080}x^8 & \frac{1}{2}x^2+\frac{1}{2}t^2-\frac{1}{144}x^6t^2-\frac{13}{10080}x^8 \\ 0 & \frac{1}{1120}x^8t^2+\frac{173}{1209600}x^{10} & 0 \\ -\frac{1}{21600}x^{10}t^2-\frac{43}{10644480}x^{12} & 0 & \frac{1}{1120}x^8t^2+\frac{173}{1209600}x^{10} \end{vmatrix}$$

$$\begin{aligned} &=(7656000x^8t^4-99792000x^6t^4+372556800x^2t^4+7185024000t^4 \\ &+2698020x^{10}t^2-34518000x^8t^2+405064800x^4t^2+8335958400x^2t^2 \\ &+245369x^{12}-2968680x^{10}+32508000x^6 \\ &+1150934400x^4)x^{16}(1080t^2+173x^2)/1946785126809600000 \end{aligned}$$

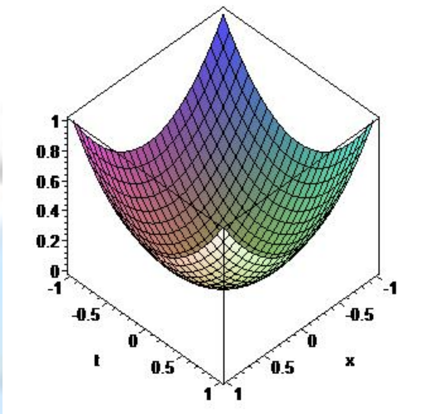
and

$$q(x,t)=\begin{vmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{1120}x^8t^2+\frac{173}{1209600}x^{10} & 0 \\ -\frac{1}{21600}x^{10}t^2-\frac{43}{10644480}x^{12} & 0 & \frac{1}{1120}x^8t^2+\frac{173}{1209600}x^{10} \end{vmatrix}$$

$$=\frac{(47520t^2+7612x^2+2464t^2x^2+215x^4)x^{16}(1080t^2+173x^2)}{64377815040000}$$

So the Multivariate Padé approximation of order $(10,2)$ for eq.(19), that is,

$$\begin{aligned}r_{10,2}(x,t) = (&7656000x^8t^4 - 99792000x^6t^4 + 372556800x^2t^4 + 7185024000t^4 + 2698020x^{10}t^2\\&-34518000x^8t^2 + 405064800x^4t^2 + 8335958400x^2t^2 + 245369x^{12} - 2968680x^{10}\\&+32508000x^6 + 1150934400x^4)/302400(47520t^2 + 7612x^2 + 2464t^2x^2 + 215x^4)\end{aligned}$$



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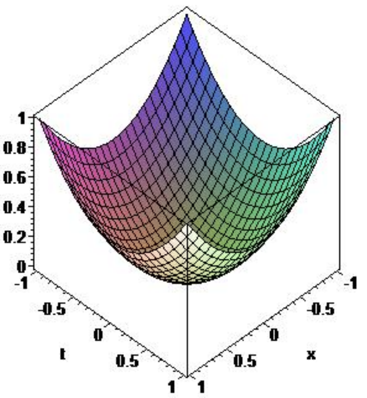

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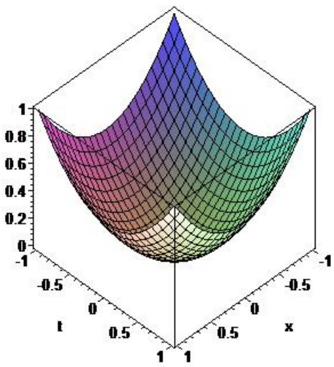

(6)

Figures **(4)** Exact solution of partial differential equation in example 2. **(5)** Adomian's decomposition solution of partial differential equation in example 2. **(6)** Multivariate Padé approximation for Adomian's decomposition solution of partial differential equation in Example 2.

Table 2. Comparison of ADM and MPA for example 2.

x	*t*	Exact solution	Approximate solution with ADM	Approximate solution with MPA	Absolute error of ADM	Absolute error of MPA
2.0	0.0	2.0	1.799749880	1.801429521	0.2002501202	0.1985704793
1.9	0.1	1.810000000	1.667678433	1.668542995	0.1423215671	0.141457005
1.8	0.2	1.640000000	1.538096667	1.538568292	0.1019033315	0.101431708
1.7	0.3	1.490000000	1.416194070	1.416465087	0.07380592976	0.073534913
1.6	0.4	1.360000000	1.305876849	1.306036780	0.05412315226	0.053963220
1.5	0.5	1.250000000	1.209948042	1.210041698	0.04005195778	0.039958302
1.4	0.6	1.160000000	1.130312651	1.130365372	0.02968734888	0.029634628
1.3	0.7	1.090000000	1.068188762	1.068216595	0.02181123746	0.021783405
1.2	0.8	1.040000000	1.024306680	1.024320203	0.01569331987	0.015679797
1.1	0.9	1.010000000	0.9990817140	0.9990876644	0.01091828564	0.0109123356
1.0	1.0	1.000000000	0.9927514167	0.9927537493	0.007248583303	0.007246250702

5. Conclusion

MPA have been successfully applied two different differential equations. The exact solutions of the examples and solutions obtained using ADM [12] were compared with MPA. The results show that MPA is very effective and convenient for solving partial differential equations.

6. References

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